could not be confirmed or denied since its effect was, in general, less than the experimental error. The only exception was at the highest eccentricity ratio investigated, $\phi = 0.96$, $\kappa = 0.93$, where the one experimental data point obtained verified the predicted dependence of terminal velocity ratio on κ.

CONCLUSION

All factors considered, the experimental data reported here provide verification of the theory developed by Chen, et al. [1], agreeing with an average deviation of 2.5%.

ACKNOWLEDGMENT

Financial support for J. A. Lescarboura during the course of this work was provided by an American Oil Foundation Doctoral Fellowship.

NOTATION

- R = inner radius of fall tube, cm.
- V_{con} = terminal velocity of cylinder when concentric to fall tube, cm./sec.
- V_{ecc} = terminal velocity of cylinder when eccentric to fall tube, cm./sec.
- ϵ = eccentricity or displacement of axis of cylinder to axis of fall tube, cm.
- κ = ratio of cylinder radius to inner radius of fall tube, dimensionless
- ϕ = eccentricity ratio, $\epsilon/[R(1-\kappa)]$, dimensionless

LITERATURE CITED

1. Chen, M. C. S., J. A. Lescarboura, and G. W. Swift, *AIChE J.*, 14, 123 (1968).
2. Lescarboura, J. A., Ph.D. thesis, Univ. Kansas, Lawrence (1967).

Thermal Instability of a Horizontal Layer of Water Near 4°C.

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When a layer of fluid is subject to an adverse temperature gradient, the system is potentially unstable. The onset of convection is indicated when the Rayleigh number exceeds its critical value. Rayleigh number is conventionally defined as

$$N_{Ra} = \frac{g\alpha(\Delta T)d^3}{\nu\kappa}$$

The critical Rayleigh number is dependent upon the boundary conditions. For the case of rigid-rigid surfaces, $(N_{Ra})_{cr}$ is found to be approximately 1,700.

The problem becomes somewhat involved if the liquid possesses a maximum density (or minimum) within the lower and upper surface temperature. A good example is that of a water layer which is subject to temperatures on either side of 4°C. The density of water increases upward from the lower surface until it reaches a maximum then it decreases. In other words, part of the liquid layer is potentially unstable while the other part is stable. Stability analysis of this problem with rigid-rigid surface has been carried out by a number of investigators (2, 3) by using the following density-temperature relationship,

$$\rho - \rho_{max} = -\rho_{max}\,\gamma(T - T_{max})^2 \qquad (1)$$

A modified Rayleigh number can be defined as

$$N_{Ra} = \frac{(2\gamma\, A\Delta T)\; g(\Delta T)d^3}{\nu\kappa} \qquad (2)$$

The critical Rayleigh number is found to be a function of parameter A which is given as

$$A = \frac{T_l - T_{max}}{T_l - T_u} \qquad (3)$$

Numerical values of $(N_{Ra})_{cr}$ vs. A are shown in Figure 1. For $A < 0.25$, the asymptotic expression of Chandrasekhar can be used

$$(N_{Ra})_{cr} \sim 1186.4\left(\frac{1}{A}\right)^4 \qquad (4)$$

It is interesting to note that Rayleigh number defined by Equation (2) is always positive because of the term of $(\Delta T)^2$. Consequently, the onset of convection is possible for both heating and cooling. Furthermore, one can show that this criterion is also consistent with physical argument. For the limiting case of $A = 1$, $T_u = T_{max}$, and there is no density inversion within the liquid layer. The critical Rayleigh number becomes

$$(N_{Ra})_{cr} = \frac{g[\gamma 2(T_l - T_{max})]\;[T_l - T_{max}]}{\nu\kappa}\, d^3 = 3{,}390$$

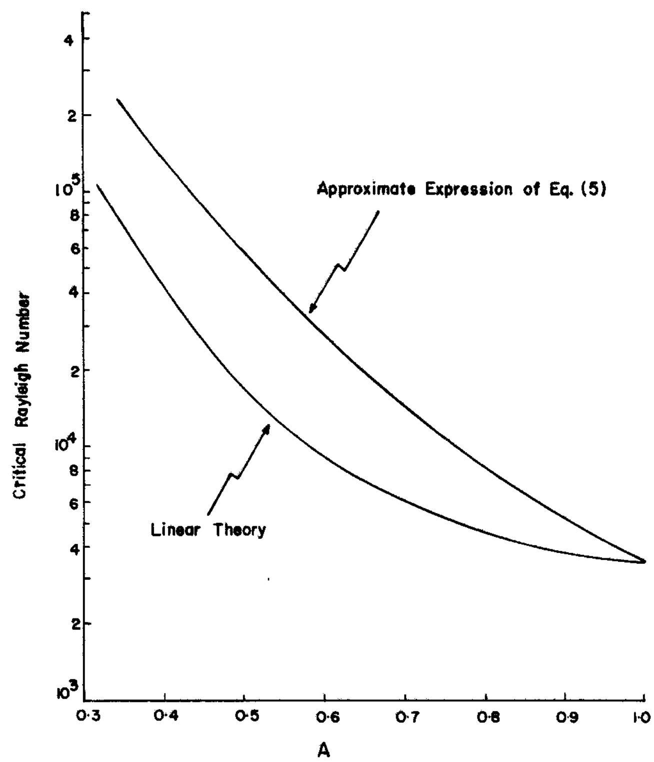

Fig. 1. Critical Rayleigh number as a function of A.

or

$$\frac{g\left[2\gamma\frac{T_l-T_{\max}}{2}\right][T_l-T_{\max}]}{\nu\kappa}d^3=1{,}695$$

The term $\left[2\gamma\frac{T_l-T_{\max}}{2}\right]$ can be considered as the average value of the thermal expansion coefficient of the liquid layer. If the results of classical Bernard problem is applied, the critical Rayleigh number is known to be 1,700 which differs by less than 1% of the value of 1,695.

The asymptotic expression of the critical Rayleigh number indicates that $N_{Ra_{cr}} \to \infty$ as $A \to 0$, or the system is always stable. This is expected since for $A = 0$, $T_l = T_{\max}$. The liquid density is a maximum at the bottom and decreases upward, a situation in which the onset of convection is obviously an impossibility.

An approximate stability criterion based on physical argument can be obtained by applying the results of the classical Bernard problem to the potentially unstable layer only. Under this assumption, the significant length and temperature differences used for the evaluation of Rayleigh number becomes $A(\Delta T)$ and Ad, respectively. The thermal expansion coefficient should be the average values, or $\left(2\gamma\frac{A\Delta T}{2}\right)$. This yields

$$N'_{Ra_{cr}}=\frac{g\left[2\gamma A\frac{\Delta T}{2}\right](A\Delta T)}{\nu\kappa}(Ad)^3=1{,}712$$

or

$$N_{Ra_{cr}}=A\frac{g(2\gamma\Delta T)\ \Delta T\ d^3}{\nu\kappa}=\frac{3{,}432}{(A)^4} \qquad (5)$$

Numerical values of the critical Rayleigh number based on this crude physical model were compared with those based on the present analysis and shown in Figure 1. The value of $N_{Ra_{cr}}$, according to Equation (5), was found to be consistently higher than the actual value, but Equation (5) does correctly predict the dependence of $(N_{Ra})_{cr}$ on the parameter A in its asymptotic form.

A comparison between the theoretical results and the recent experimental observation of Boger and Westwater (*1*), is made and summarized in Table 1. The prediction of the onset of convection was correct in all cases. In this comparison, the presence of the solid phase (ice) was neglected. The depth of the liquid layer is taken as $D - \epsilon_M$ where D and ϵ_M are the height of the test chambers and the measured ice thickness, respectively. The surface temperatures were 0°C. and T_a (in terms of the notation in reference *1*). The Rayleigh number was evaluated according to Equation (2). All the physical properties were evaluated at the average temperatures (that is, $T_a/2$), the value of γ is taken as 8×10^{-6} (0°C.)$^{-2}$ which was obtained by fitting the density temperature data of water according to Equation (1). For cases where $T_a >> 8$°C., the term $(2\gamma A\Delta T)$ is replaced by the actual value of thermal expansion of water at T_a since the form of expression of Equation (1) no longer holds for $T > 8$°C.

TABLE 1. COMPARISONS BETWEEN THE PREDICTIONS BASED ON THIS WORK AND THE EXPERIMENTAL OBSERVATIONS (*1*)

Run no.	T_l*	T_u*	d≠	A	N_{Ra}	$N_{Ra_{cr}}$	Convections Prediction	Convections Experimental observations
10B	0	75.2	1.798	0.05319	2.5719×10^6	1.4825×10^8	no	no
11A	0	62.3	2.293	0.0642	4.6887×10^6	6.9861×10^7	no	no
11E	0	61.3	2.363	0.0653	4.2990×10^6	6.5272×10^7	no	no
12D	0	8.0	0.397	0.5	1.411×10^3	1.868×10^4	no	no
12H	8.2	0	0.445	0.5122	2.1436×10^3	1.63×10^4	no	no
12R	8.9	0	0.523	0.5506	4.4565×10^3	1.22×10^4	no	no
12P	9.8	0	0.545	0.5918	6.6640×10^3	9.4×10^3	no	no
12Z	9.3	0	0.742	0.5699	1.3387×10^4	1.09×10^4	yes	yes
12Q	9.8	0	0.795	0.5918	1.8996×10^4	9.4×10^3	yes	yes
12N	12.2	0	0.976	0.6721	6.2082×10^4	6.3×10^3	yes	yes
12J	14.5	0	2.353	0.7241	1.3139×10^5	5.38×10^3	yes	yes
12M	28.5	0	3.847	0.8596	4.153×10^7	3.85×10^3	yes	yes
12V	24.9	0	4.054	0.8394	3.3882×10^7	4.1×10^3	yes	yes
12K	27.4	0	3.816	0.854	3.6495×10^7	4.0×10^3	yes	yes

* in. °C. ≠ in. cm.

ACKNOWLEDGMENT

This work is performed under Grants No. DA-AMC-27-021-65-G16 and DA-AMC-27-021-G20, U. S. Army Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

NOTATION

- A = temperature difference ratio defined by Equation (3)
- d = height of fluid layer
- g = gravitational acceleration
- N_{Ra} = Rayleigh number defined by Equation (2)
- $N_{Ra_{cr}}$ = critical Rayleigh number
- $T_{\max}$ = maximum density temperature
- T_l = temperature at lower surface of fluid layer
- T_u = temperature at upper surface of fluid layer
- γ = coefficient in density temperature relationship given by Equation (1)
- ρ = density
- $\rho_{\max}$ = maximum density
- κ = thermal diffusivity
- ν = kinematic viscosity

LITERATURE CITED

1. Boger, D. V., and J. W. Westwater, *J. Heat Transfer, Ser. C.*, **89**, 81 (1967).
2. Debler, W. R., *J. Fluid Mech.*, **24**, 165 (1966).
3. Vernois, G., *Astrophysical J.*, **137**, 641 (1963).